

A CHARACTERIZATION OF ISOMETRIES OF $CAT(0)$ -SPACE AS MAPS PRESERVING DIAGONAL TUBE

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ABSTRACT. We give positive answers for questions by Berestovskii. Namely, we prove that every bijection of locally compact geodesically complete and connected at infinity $CAT(0)$ -space X onto itself preserving some fixed distance or satellite relations is an isometry of this space. The proof of this theorem is based on another result stated by Berestovskii as a problem: the metric of the space X may be recovered from its diagonal tube corresponding to an arbitrary number $r > 0$.

INTRODUCTION

V.N.Berestovskii in [B1] has established following characterisation for isometries of Aleksandrov spaces of curvature negatively bounded above:

Theorem A (Berestovskii). *Let X be a locally compact geodesically complete $CAT(\kappa)$ -space, $\kappa < 0$, in which all spheres are path connected. Then every bijection f of X onto itself such that both f and f^{-1} map any closed ball of some fixed radius $r > 0$ onto some ball of radius r is isometry of X .*

The key point of the proof is

Theorem B (Berestovskii). *Let X be as in Theorem A and $V \subset X \times X$ be its diagonal tube corresponding to a number $r > 0$. Then the metric of X is uniquely determined by V .*

A *diagonal tube* V of a metric space X corresponding to $r > 0$ is by definition a set

$$V := \{(x, y) \in X \times X \mid |xy| \leq r\} \subset X \times X,$$

where $|xy|$ is a distance between points $x, y \in X$. For a set V we put

$$\partial V := \{(x, y) \in X \times X \mid |xy| = r\} \subset X \times X,$$

and

$$\text{Int}V := \{(x, y) \in X \times X \mid |xy| < r\} \subset X \times X.$$

A question whether analogous theorems are true in the case $\kappa = 0$ is raised in [B1]. We give an affirmative answer to this question. Namely we prove

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Theorem 0.1. *Let (X, d) be a locally compact geodesically complete and connected at infinity $CAT(0)$ -space, $f : X \rightarrow X$ — bijection and $V \subset X \times X$ be diagonal tube of space X corresponding to a number $r > 0$. Set a map $\phi := f \times f : X \times X \rightarrow X \times X$ by $\phi(x_1, x_2) := (f(x_1), f(x_2))$. Then following statements are equivalent:*

- (1) $\phi(V) = V$,
- (2) $\phi(\partial V) = \partial V$,
- (3) $\phi(\text{Int } V) = \text{Int } V$,
- (4) f is an isometry of X onto itself.

Remark 0.2. When we say that a map ϕ moves V onto V this means that for every pair $x, y \in X$ with $d(x, y) \leq r$ inequality $d(\phi(x), \phi(y)) \leq r$ holds, but not necessarily $d(\phi(x), \phi(y)) = d(x, y)$. The last equality is the target of the theorem.

Remark 0.3. Connectedness at infinity of space X means that the complement $X \setminus B$ of every metric ball $B \subset X$ is path connected. It was shown in [B1] that connectedness at infinity is equivalent to the condition of path connectedness for every sphere in X .

Following [B1] we base the proof on the uniqueness of the metric on X with prescribed properties and given diagonal tube. We formulate this statement as following.

Theorem 0.4. *Let the space (X, d) be as in theorem 0.1 and the number $r > 0$ is fixed. Then every metric d' on X such that (X, d') is locally compact, geodesically complete $CAT(0)$ -space, coincides with d : for any $x, y \in X$ $d'(x, y) = d(x, y)$, iff any one of following equivalent conditions hold:*

- (1) $\forall x, y \in X \quad d(x, y) \leq r \Leftrightarrow d'(x, y) \leq r$;
- (2) $\forall x, y \in X \quad d(x, y) = r \Leftrightarrow d'(x, y) = r$;
- (3) $\forall x, y \in X \quad d(x, y) < r \Leftrightarrow d'(x, y) < r$.

Theorem 0.1 is direct consequence of theorem 0.4: it suffices to take the metric $d' = f^*d$:

$$d'(x, y) = d(f(x), f(y)),$$

and apply equality $f^*d(x, y) := d(x, y)$ as claim of theorem 0.4.

For simplicity, from now on we will assume $r = 1$, but conveniently, the notation r will be kept in some notions, such as r -sequences etc.

The author have considered a case of Busemann spaces ([Bus]), i.e. locally compact complete metric spaces with intrinsic metric and properties of nonbranching and local extendability of minimizing segments in earlier paper ([An]). It was shown in [B2] that every $CAT(0)$ -space which is a Busemann space is really a Riemannian manifold with continuous metric tensor relatively distance coordinates. Thus the connectedness at infinity of a space X is equivalent to the estimation $n > 1$ for its topological dimension $n := \text{TopDim}(X)$

in this case. The proof of the theorem 0.4 in the present paper is based on methods developed in [An] with necessary modification.

Problems resolved in the paper are conjugated with the question suggested by A.D.Aleksandrov in 1960-s:

- Under what conditions is a map of a metric space into itself preserving a fixed distance (for example, distance 1) an isometry of this metric space?

A number of similar questions are partially resolved by Aleksandrov himself. For example, several theorems presenting sufficient conditions for map of classical spaces of constant curvature preserving congruence to be isometry are proved in [Al]. Beckman and Quarles have proved the version of theorem 0.1 for maps of Euclidean spaces in [BQ], Kuzminykh — for Lobachevskii space in [K].

The paper consists of three sections. The first section is introductory and contains a list of basic notions and facts used by author as preliminaries and the concept of parallel-equivalence for r -sequences. We introduce two types of r -sequences according to behavior of corresponding classes of parallel-equivalence. In the rest sections we show that the metric of any geodesic in X can be recovered from V . In the Section 2 we study geodesics bounding flat strip, and in Section 3 geodesics of strictly rank one. Combining two mentioned situations we get whole proof of Theorem 0.4.

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1. PRELIMINARIES

1.1. $CAT(\kappa)$ -spaces. Main definitions and properties of Aleksandrov spaces with curvature bounded above and so called $CAT(\kappa)$ -spaces may be found in [Ba], [BH] or [Bu].

Let (X, d) be a metric space. The distance d between points $x, y \in X$ conveniently will be denoted as $|xy| := d(x, y)$. An open ball of radius ρ centered at point $x \in X$ is denoted as $B(x, \rho)$, corresponding closed ball — as $\overline{B}(x, \rho)$, the boundary sphere — as $S(x, \rho)$. For any subset $A \subset X$ and any $\epsilon > 0$, the ϵ -neighborhood of A is

$$N_\epsilon(A) := \{x \in X \mid d(x, a) < \epsilon \text{ for some } a \in A\}.$$

For any two closed subsets $A, B \subset X$, the *Hausdorff distance* between A and B is

$$d_H(A, B) := \inf\{\epsilon \mid A \subset N_\epsilon(B), B \subset N_\epsilon(A)\};$$

$d_H(A, B)$ is defined to be ∞ if there is no $\epsilon > 0$ with $A \subset N_\epsilon(B)$ nor $B \subset N_\epsilon(A)$.

A *geodesic* in X is a continuous map $c : I \rightarrow X$ from an interval $I \subset \mathbb{R}$ into X , such that for any point $t \in I$ there exists a neighborhood U of t with $d(c(s_1), c(s_2)) = |s_1 - s_2|$ for all $s_1, s_2 \in U$. If one can take $U = I$, then such geodesic is said to be *minimizing*. The image of a geodesic or a minimizing will also be called a geodesic or a minimizing. When I is a closed interval $[a, b] \subset \mathbb{R}$, we say that c is a *geodesic segment* of length $b - a$ and c *connects* $c(a)$ and $c(b)$. If $I = \mathbb{R}$, we say that c is a complete geodesic. A metric space X is called a *geodesic metric space* if for any two points $x, y \in X$ there is a minimal geodesic segment connecting them. A geodesic metric space is called *geodesically complete* if every geodesic segment is contained in some complete geodesic (not necessarily unique).

It follows from Hopf-Rinow theorem (cf. [Bu, Ch.1, Theorem 2.3]) that geodesically complete locally compact space X is *proper* or *finitely compact*, i.e. every its closed ball is compact.

For $\kappa \in \mathbb{R}$ we let M_κ be the *model* (i.e. complete simply connected) surface of constant curvature κ , $D(\kappa)$ be its *diameter*, that is, $D(\kappa) = \infty$ if $\kappa \leq 0$ and $D(\kappa) = \pi/\sqrt{\kappa}$ if $\kappa > 0$. The metric of M_κ is denoted as d_κ .

A *triangle* in X is the union of three minimizing segments $c_i : [a_i, b_i] \rightarrow X$, $i = \overline{1, 3}$, called *sides* of triangle, pairwise connecting three points $x_i, i = \overline{1, 3}$ which we call *vertices* of triangle.

For a collection $\Delta := (x_1, x_2, x_3)$ of points in X the *comparison triangle* $\overline{\Delta} \subset M_\kappa$ has vertices $\overline{x}_1, \overline{x}_2$ and \overline{x}_3 such that $d_\kappa(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$, $i, j = \overline{1, 3}$. A comparison triangle exists and is unique up to isometry if the perimeter

$$P(\Delta) := d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3) < 2D(\kappa).$$

If c_i are sides of Δ , we denote \overline{c}_i sides of $\overline{\Delta}$.

A point $\overline{x} \in \overline{\Delta}$ corresponds to a point x of triangle Δ if there is some i and some $t_i \in [a_i, b_i]$ with $x = c_i(t_i)$ and $\overline{x} = \overline{c}_i(t_i)$.

Let $\kappa \in \mathbb{R}$. A complete metric space X is called a *CAT(κ)-space* if

- (i) Every two points $x, y \in X$ with $d(x, y) < D(\kappa)$ are connected by a minimizing;
- (ii) For any triangle Δ in X with perimeter less than $2D(\kappa)$ and any two points $x, y \in \Delta$, the inequality $d(x, y) \leq d(\overline{x}, \overline{y})$ holds, where \overline{x} and \overline{y} are the points of the comparison triangle $\overline{\Delta} \subset M_\kappa$ corresponding to x and y respectively.

Every geodesic $c : \mathbb{R} \rightarrow X$ in CAT(0)-space (X, d) has two opposite directions defined by its rays $c|_{(-\infty, 0]}$ and $c|_{[0, +\infty)}$. Geodesic rays $c_i : [0, \infty) \rightarrow X, i = \overline{1, 2}$ are called *asymptotic* if Hausdorff distance between them is finite: $d_H(c_1, c_2) < \infty$. A relation *to be asymptotic* on a set of all rays in X is equivalence. Equivalence classes called *ideal points* (or *points at infinity*) form *geometric boundary* (*boundary at infinity*) $\partial_\infty X$ of X . Complete geodesics c_1 and c_2 are *asymptotic* if they contain asymptotic rays.

Complete geodesics c_1 and c_2 are called *parallel* if their Hausdorff distance $\text{Hd}(c_1, c_2)$ is finite. In this case they are asymptotic in both directions and bound a *flat strip*, i.e. subset isometric to a strip in Euclidean plane.

Definition 1.1. We say that a geodesic c *virtually bounds a flat strip* if there is a finite sequence $c = c_0, c_1, \dots, c_n$ of geodesics such that for every $1 \leq i \leq n$ geodesics c_{i-1} and c_i are asymptotic and c_n is a boundary geodesic of some flat strip. In particular every line bounding the flat strip also bounds it virtually. Otherwise we say that c *has strictly rank one*.

1.2. Spaces of directions. A *pseudo-metric* on a set Σ is a function $d : \Sigma \times \Sigma \rightarrow [0, \infty)$ that is symmetric and satisfies the triangle inequality. If (Σ, d) is a pseudo-metric space, then we get a metric space (Σ^*, d^*) by letting Σ^* be the set of maximal zero diameter subsets and setting $d^*(S_1, S_2) := d(s_1, s_2)$ for any $s_i \in S_i$. (X^*, d^*) is called the metric space *associated* to the pseudo-metric d .

Given two geodesic segments $c = [x\eta]$ and $d = [x\zeta]$ with common vertex x in $CAT(\kappa)$ -space X , $\kappa \in \mathbb{R}$ with lengths less than $D(\kappa)$, we have a well-defined function $\angle_x(c, d) = \lim_{t \rightarrow 0} \widetilde{\angle}_x(c(t), d(t))$, where $\widetilde{\angle}_x(y, z)$ denotes *the comparison angle*, i.e. angle at the vertex \bar{x} of comparison triangle $\bar{x}\bar{y}\bar{z}$ at the Euclidean plane for triangle xyz . The function \angle_x defines a pseudo-metric on a set of all geodesic segments beginning from x . The metric space associated to the pseudo-metric \angle_x is denoted by $\Sigma_x^* X$. Its metric completion $\Sigma_x X$ is called *the space of directions* at x . If a space X is geodesically complete, then we have $\Sigma_x^* X = \Sigma_x X$ for any $x \in X$.

Theorem 1.2 (Nikolaev [N]). *Let X be a $CAT(\kappa)$ -space and $p \in X$. Then $\Sigma_p X$ is a $CAT(1)$ -space.*

1.3. Diagonal tube and satellite relations. From now on we assume that (X, d) is a locally compact geodesically complete connected at infinity $CAT(0)$ -space (*Hadamard space*) and $\overline{X} = X \cup \partial_\infty X$ be its geometric closure. The metric d of $CAT(0)$ -space is convex function in following sense. For every two geodesics $c_1, c_2 : \mathbb{R} \rightarrow X$ the function $d(c_1(t), c_2(t))$ is convex.

For two points $y, z \in \overline{X}$ notation $[yz]$ means:

- connecting them geodesic segment, if both lies in X , or
- connecting them geodesic ray if one of them lies in X , and another (order is ignored) — in $\partial_\infty X$, or
- any complete geodesic connecting y and z in the case when $y, z \in \partial_\infty X$, if such geodesic does exist. When the geodesic is not unique, we will detect it by additional features.

We will identify every geodesic as a map of real interval into X with its image and use the same notation.

For a number $r > 0$ V will be *the diagonal tube* corresponding it. Recall that we have assumed $r = 1$, hence

$$V := \{(x, y) \in X \times X \mid |xy| \leq 1\}.$$

When we say that diagonal tube V defines metric d , this means that every metric d' on X satisfying conditions of theorem 0.4 with the same diagonal tube V coincides with d . We will consider d as initial metric and d' as trial metric for which we need to show that $d' = d$. Below d' will always be trial metric satisfying conditions of theorem 0.4 and with the same diagonal tube V .

Also, sometimes we will use the terminology, such as following: "diagonal tube defines ..." or " V allows to recover..." etc. Opinions of such type will mean that some object, value or some property of object is the same for metric d and any trial metric d' . In particular this holds when mentioned object admits description in terms of V .

First, we note that relations nV , $\text{int}(nV)$ and $\partial(nV)$ are defined for all $n \in \mathbb{N}$ simultaneously with V by equalities

$$nV := \{(x, y) \in X \times X \mid |xy| \leq n\}$$

$$\partial(nV) := \{(x, y) \in X \times X \mid |xy| = n\}$$

and

$$\text{Int}(nV) := nV \setminus \partial(nV).$$

Lemma 1.3. *If two metrics d and d' on X as in theorem 0.4 have common diagonal tube V , then they have also common relations nV , $\partial(nV)$ and $\text{Int}(nV)$ for all $n \in \mathbb{N}$.*

Proof. Pair $(x, y) \in 2V$ iff there exists a point $z \in X$ such that $(x, z), (z, y) \in V$. Inductively, $(x, y) \in nV$ iff there exists $z \in X$ such that $(x, z) \in (n-1)V$ and $(z, y) \in V$. We have $(x, y) \in \partial(2V)$ iff there exists unique point z with $(x, z), (z, y) \in V$. In this case (x, z) and $(z, y) \in \partial V$ and z is a midpoint of segment $[xy]$. Moreover, pair $(x, z) \in \partial V$ iff there exists a pair $(x, y) \in \partial(2V)$ such that z is a midpoint of $[xy]$. Inductively one may define relations $\partial(nV)$. Relations $\text{Int}(nV)$ are by definition

$$\text{Int}(nV) = nV \setminus \partial(nV).$$

□

Consequently metrics d and d' with common V have common open, closed balls and spheres with integer radii.

Lemma 1.4. *If two metrics d and d' on X as in theorem 0.4 have common any one of three relations V , ∂V and $\text{Int} V$, then they have common another two relations.*

Proof. It was shown in Lemma 1.3 that V defines ∂V and $\text{Int}(V)$.

Assume that d and d' have common boundary relation ∂V . Arguments similar to those used in the proof of Lemma 1.3 show that they have common diagonal tube $2V$ as well. In particular, metrics d and d' have common all balls $\overline{B}(x, 2n)$ of even radii and consequently all spheres $S(x, 1)$ and $S(x, 2n)$ with $x \in X$. Then by the connectedness at infinity of X we have $(x, y) \in \text{Int}V$ iff $S(x, 1) \cap S(y, 1) \neq \emptyset$ and $S(x, 2) \cap S(y, 1) = \emptyset$. This defines $\text{Int}V$ by already defined relations. V is defined as $V = \text{Int}V \cup \partial V$.

Let now d and d' have common $\text{Int}V$. Then for all $n \in \mathbb{N}$ relations $\text{Int}(nV)$ are also common for d and d' . Hence from geodesic completeness of X we have that $(x, y) \in \partial V$ iff $(x, y) \notin \text{Int}V$ and $B(y, 1) \subset B(x, 2)$. Indeed, if $1 < d(x, y) < 2$, then we may include geodesic segment $[xy]$ into segment $[xz]$ of length $d(x, z) = \frac{1}{2}(3 + d(x, y))$, and $z \in B(y, 1) \setminus B(x, 2)$. This defines ∂V and hence V in terms of $\text{Int}V$. \square

Consequently for the proof of theorem 0.4 we need only to prove that the metric d may be recovered from diagonal tube V itself.

1.4. r -sequences. The main tool of [B1], called r -sequence was defined in terms of relations above. We give another definition here. r -sequence in X is by definition a homothety $\mathbb{Z} \rightarrow X$ with coefficient r , that is an integral parametrized sequence $\{x_z\}_{z \in \mathbb{Z}} \subset X$ of points in X , such that for all $z_1, z_2 \in \mathbb{Z}$ equalities $|x_{z_1} x_{z_2}| = r|z_2 - z_1|$ hold. When we put $r = 1$, the last equation becomes written $|x_{z_1} x_{z_2}| = |z_2 - z_1|$ and we consider r -sequence as isometric map $\mathbb{Z} \rightarrow X$.

It immediately follows from lemmas 1.3 and 1.4 and geometry of space (X, d) that:

- for every r -sequence there is unique containing it geodesic in (X, d) and geodesic in (X, d') . A priori these geodesics may be different. If they coincide, we will say that the incidence relation on this geodesic is detected by V .
- for any trial metric d' , sequence $\{x_z\}_{z \in \mathbb{Z}}$ is r -sequence with respect to the metric d iff it is r -sequence with respect to d' , and
- V allows us to reveal, given two r -sequences $\{x_z\}_{z \in \mathbb{Z}}$ and $\{y_z\}_{z \in \mathbb{Z}}$ whether geodesics in (X, d) containing them are asymptotic in any direction.

The segment of r -sequence $\{x_z\}_{z \in \mathbb{Z}} \subset X$ between points x_z and x_{z+k} will be denoted as $[x_z, \dots, x_{z+k}]_r$, ideal points, defined by r -sequence $\{x_z\}_{z \in \mathbb{Z}}$, — as $x_{+\infty}$ and $x_{-\infty}$.

1.5. Metric transfer. We will use an effective trick proposed in [B1], — a map $R_{c_1 c_2}$ from a geodesic to its asymptotic one by means of *Busemann functions*. Let $c_i : \mathbb{R} \rightarrow X, i = \overline{1, 2}$ be two asymptotic geodesics such that

$c_1(+\infty) = c_2(+\infty)$. Let β_ξ be some Busemann function corresponding to an ideal point $\xi = c_i(+\infty)$:

$$(1.5) \quad \beta_\xi(x) := \lim_{t \rightarrow \infty} (|x c(t)| - t).$$

Level sets of function β_ξ are called *horospheres*, sublevels — *horoballs*. We will distinguish open horoballs defined by strict inequality $\beta_\xi(x) < b_0$ and horoballs $\beta_\xi(x) \leq b_0$.

The horosphere centered in the end $\xi := c(+\infty) \in \partial_\infty X$ of geodesic $c : \mathbb{R} \rightarrow X$ containing a point $z \in X$, i.e. level set

$$\{x \in X \mid \beta_\xi(x) = \beta_\xi(z)\}$$

of Busemann function $\beta_\xi(x)$ (1.5) will be denoted as $\mathcal{HS}_{\xi,z}$, corresponding closed horoball as $\mathcal{HB}_{\xi,z}$. An open horoball is $hb_{\xi,z} := \mathcal{HB}_{\xi,z} \setminus \mathcal{HS}_{\xi,z}$.

Horoballs and horospheres admits following definitions in terms of $\text{Int}V$ (and hence in terms of V). Let $\{x_z\}_{z \in \mathbb{Z}}$ be r -sequence with $\xi := x_{+\infty} = c_i(+\infty)$ and given point x_0 . Then open horoball hb_{ξ,x_0} is

$$hb_{\xi,x_0} = \bigcup_{n=1}^{\infty} B(x_n, n).$$

For a point $y \in X$ we have $y \in \mathcal{HS}_{\xi,x_0}$ iff $y \notin hb_{\xi,x_0}$ and $B(y, 1) \subset hb_{\xi,x_{-1}}$. Finally, $\mathcal{HB}_{\xi,x_0} = hb_{\xi,x_0} \cup \mathcal{HS}_{\xi,x_0}$.

There exist length parametrisations of geodesics c_i , such that

$$(1.6) \quad \forall t \in \mathbb{R} \quad \beta_\xi(c_1(t)) = \beta_\xi(c_2(t)).$$

We set $R_{c_1 c_2}(c_1(t)) := c_2(t)$. A map $R_{c_1 c_2}$ defined above is independent on a choice of Busemann function β_ξ centered in $c_i(+\infty)$ and on parametrisations with the property (1.6).

It was shown in [B1] that whenever the relation of incidence of points in lines c_i is detected, then relation V defines every map $R_{c_1 c_2}$ in metric space (X, d) itself. This means that whenever geodesics c_1 and c_2 are also geodesics with respect to trial metric d' , then the image of any point $c_1(t)$ under the map $R_{c_1 c_2}$ is independent on the choice of metric d or d' .

Definition 1.7. We say that two geodesics a and b are *connected by the asymptotic chain* if there is a finite sequence

$$(1.8) \quad a = a_0, a_1, \dots, a_n = b$$

of geodesics such that for every $1 \leq i \leq n$ geodesics a_{i-1} and a_i are asymptotic in some direction.

Example 1.9. If geodesic a virtually bounds flat strip, it is connected by asymptotic chain with some geodesic b lying in the boundary of flat strip.

Lemma 1.10. *Let geodesics a and b be connected by the asymptotic chain (1.8) such that for all $i = \overline{0, n}$ relation of incidence in geodesics a_i is detected by V . Then if distances d and d' coincide along a , i.e. for every pair $t_1, t_2 \in \mathbb{R}$ equality $d(a(t_1), a(t_2)) = d'(a(t_1), a(t_2))$ holds, then metrics coincide along b as well.*

Proof. The claim may be proved by a multiple transfer of a metric from a geodesic to its asymptotic one in the asymptotic chain connecting a and b . \square

1.6. Parallel equivalence.

Definition 1.11. Let $(\{x_z\}_{z \in \mathbb{Z}}, \{y_z\}_{z \in \mathbb{Z}})$ be a pair of r -sequences in X . We say that they are *parallel-equivalent*, iff Hausdorff distance d_H between sets $\{x_z\}_{z \in \mathbb{Z}}$ and $\{y_z\}_{z \in \mathbb{Z}}$ is finite:

$$d_H(\{x_z\}_{z \in \mathbb{Z}}, \{y_z\}_{z \in \mathbb{Z}}) < +\infty.$$

We will use the notation $\{x_z\}_{z \in \mathbb{Z}} \parallel \{y_z\}_{z \in \mathbb{Z}}$ for this relation. It is obviously a really equivalence on a set of all r -sequences in X . In fact every r -sequence belongs to a unique geodesic in X , and $\{x_z\}_{z \in \mathbb{Z}} \parallel \{y_z\}_{z \in \mathbb{Z}}$ iff geodesics containing $\{x_z\}_{z \in \mathbb{Z}}$ and $\{y_z\}_{z \in \mathbb{Z}}$ are parallel or coincide.

Lemma 1.12. *Property "to be parallel-equivalent" can be revealed by V .*

Proof. $\{x_z\}_{z \in \mathbb{Z}} \parallel \{y_z\}_{z \in \mathbb{Z}}$ iff there exists $k \in \mathbb{N}$ such that for all $z \in \mathbb{Z}$ $(x_z, y_z) \in kV$. This completes the proof. \square

Definition 1.13. r -sequence is called *r -sequence of rank 1*, if there exists unique geodesic c in X containing all r -sequences parallel-equivalent to it. c has not different parallel geodesics in X and does not bound a flat strip in this case. It was shown in [B1] that the incidence relation is detected by the diagonal tube in this case. In opposite case we say that r -sequence is *of higher rank* or has *rank over 1*. Geodesic containing r -sequence of higher rank bounds some flat strip.

Proposition 1.14. *A property of an arbitrary r -sequence $(\{x_z\}_{z \in \mathbb{Z}}$ to be of rank 1 or of higher rank is determined by relation V .*

Proof. $\{x_z\}_{z \in \mathbb{Z}}$ is of higher rank iff there exists r -sequence $\{y_z\}_{z \in \mathbb{Z}}$ parallel-equivalent to it and not equal to $\{x_{z \pm 1}\}_{z \in \mathbb{Z}}$, such that $|x_0 y_0| = 1$. Since the relation ∂V is determined by V (cf Lemma 1.3), the proposition is proved. \square

The plan of further consideration is following. We will examine every single geodesic in accordance with its rank. It will be shown that in any case the incidence relation and the metric of a geodesic may be recovered from a diagonal tube V . First we consider a case of geodesic bounding a flat strip.

2. THE GEODESIC BOUNDING FLAT STRIP

2.1. Splitting of parallel-equivalence class. Let $c : (-\infty, +\infty) \rightarrow X$ be a geodesic connecting ideal points $\xi_- := c(-\infty)$ and $\xi_+ := c(+\infty)$ and passing through the point $x_0 = c(0)$. For $x \in X$ and $\xi, \eta \in \partial_\infty X$, angle at point x between rays $[x\xi]$ and $[x\eta]$ is denoted as $\angle_x(\xi, \eta)$.

The following lemma is well-known (cf. [DOZ, Lemma 5] for example).

Lemma 2.1. *Set*

$$C := \{y \in X \mid \angle_y(\xi_-, \xi_+) = \pi\}.$$

Then

- (1) $\mathcal{HS}_{\xi_-, x_0} \cap \mathcal{HS}_{\xi_+, x_0} = \mathcal{HB}_{\xi_-, x_0} \cap \mathcal{HB}_{\xi_+, x_0}$,
- (2) C is a union of all geodesics parallel to c ,
- (3) C is a closed convex set in X and
- (4) C splits as a product $C = C' \times c$, where $C' := \mathcal{HS}_{\xi_-, x_0} \cap \mathcal{HS}_{\xi_+, x_0}$ is a closed convex subset.

Hence the set C of all points of all r -sequences parallel-equivalent to given r -sequence $\{x_z\}_{z \in \mathbb{Z}} \subset X$ splits as $C = C' \times c$. The set C' has more than one point iff the r -sequence $\{x_z\}_{z \in \mathbb{Z}} \subset X$ is of higher rank.

Remark 2.2. Given arbitrary r -sequence $\{x'_z\}_{z \in \mathbb{Z}}$ parallel-equivalent to r -sequence $\{x_z\}_{z \in \mathbb{Z}} \subset X$, denote as $C'(x'_z)$ the fiber of a product $C = C' \times c$ containing a point x'_z . Since horospheres $\mathcal{HS}_{\xi_-, x_0}$ and $\mathcal{HS}_{\xi_+, x_0}$ may be defined in terms of V , the *horizontal structure* of splitting is also determined by V . In other words any trial metric d' gives rise to the same set of fibers of type $C'(x)$, which we will call *horizontal sections* of C .

As well, diagonal tube V allows to restore the order of horizontal sections of C as following. We say that a section $C'(x)$ of the set C *lays below* of a section $C'(y)$ if a ray $[x\xi_+]$ intersects $C'(y)$. We have $\mathcal{HB}_{\xi_-, x} \cap \mathcal{HB}_{\xi_+, y} = \emptyset$ in this case. It is left to restore distances between horizontal sections of C and incidence of points of c for completing the case.

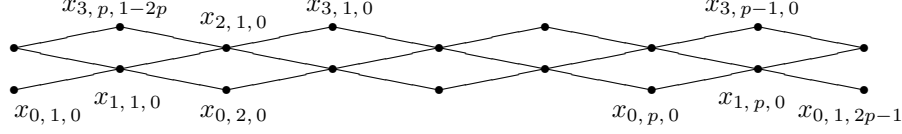
2.2. Tapes.

Definition 2.3. We say that $4p$ ($p \in \mathbb{N}$) pairwise parallel-equivalent r -sequences

$$(2.4) \quad \{x_{i,j}; z\}_{z \in \mathbb{Z}}, i = \overline{0, 3}, j = \overline{1, p}$$

form p -tape, if following $4p + 4$ points

$$\begin{aligned} & x_{i,1,0}, \dots, x_{i,p,0}, i = \overline{0, 3} \\ & x_{0,1,2p-1}, x_{2,p,1-2p}, x_{3,p-1,1-2p}, x_{3,p,1-2p} \end{aligned}$$

FIGURE 1. p -tape

in addition satisfy the system of $2p$ relations

$$(2.5) \quad \left\{ \begin{array}{l} [x_{0,1,0}, x_{1,1,0}, x_{2,1,0}, x_{3,1,0}]_r \\ \dots \\ [x_{0,p,0}, x_{1,p,0}, x_{2,p,0}, x_{3,p,0}]_r \\ [x_{0,2,0}, x_{1,1,0}, x_{2,p,1-2p}, x_{3,p-2,1-2p}]_r \\ [x_{0,3,0}, x_{1,2,0}, x_{2,1,0}, x_{3,p-1,1-2p}]_r \\ [x_{0,4,0}, x_{1,3,0}, x_{2,2,0}, x_{3,1,0}]_r \\ \dots \\ [x_{0,p,0}, x_{1,p-1,0}, x_{2,p-2,0}, x_{3,p-3,0}]_r \\ [x_{0,1,2p-1}, x_{1,p,0}, x_{2,p-1,0}, x_{3,p-2,0}]_r \end{array} \right.$$

presenting segments of r -sequences.

Since r -sequences admit definition in terms of V , notion of p -tape is independent on the choice of metric d or trial metric d' . A fragment of a p -tape is shown at figure 1. Here only a part of points of r -sequences forming p -tape is picked out. The main idea is that a segment between points $x_{0,1,0}$ and $x_{0,1,2p-1}$ of the same r -sequence contains $2(p-1)$ points of this r -sequence dividing corresponding geodesical segment by $2p-1$ equal parts, and simultaneously marked segment contains $p-1$ points of kind $x_{0,j,0}$ $j = \overline{2, p}$, dividing it by p equal segments. Hence all points of r -sequences $\{x_{0,j,z}\}_{z \in \mathbb{Z}}$ at segment $[x_{0,1,0} x_{0,1,2p-1}]$ divide it by $p(2p-1)$ equal parts, and in particular segment $[x_{0,1,0} x_{0,1,1}]$ is divided by points $x_{0,p,3-2p}, x_{0,p-1,5-2p}, \dots, x_{0,2,-1}$ by p equal segments.

Now the plan of restoring the metric of geodesic of higher rank is following. Given r -sequence $\{x_z\}_{z \in \mathbb{Z}}$ of higher rank, geodesic $c : (-\infty, +\infty) \rightarrow X$ containing it spans a flat strip (not unique in general). For sufficiently great P for any $p > P$, p -tape defined by a family of r -sequences of type (2.4) with $x_{0,1,z} = x_z$ for all $z \in \mathbb{Z}$, does exist. Points of all such r -sequences $x_{0,j,z}$ for every possible p -tapes with $p > P$ cover a set of rational points on considered geodesic.

As it was mentioned earlier, horospheres and horoballs of $CAT(0)$ -space X define horisontal sections of a set C . Hence the metric of a term \mathbb{R} in the

splitting $C = C' \times \mathbb{R}$ of the set C above may be recovered from the relation V .

It remains to show, that the incidence of points on c is restorable by V . It is already done for rational points. The limiting procedure using balls is possible for irrationals.

2.3. Recovery of the metric of geodesic bounding flat strip.

Lemma 2.6. *Every p -tape in X is contained in a flat strip.*

Proof. Let p -tape be defined by r -sequences (2.4). Consider a flat strip F spanned by parallel geodesics c and c' , containing correspondingly r -sequences $\{x_{0,1,z}\}_{z \in \mathbb{Z}}$ and $\{x_{3,1,z}\}_{z \in \mathbb{Z}}$. r -sequences $\{x_{1,1,z}\}_{z \in \mathbb{Z}}$ and $\{x_{2,1,z}\}_{z \in \mathbb{Z}}$ are also contained in F .

We will show that F really contains all r -sequences in (2.4). Consider points

$$(2.7) \quad x_{0,2,0}, \quad x_{1,1,0}, \quad x_{2,1,0}, \quad x_{3,1,0}, \quad x_{2,2,0}, \quad x_{1,2,0}.$$

As a corollary of (2.5) we have relations:

$$[x_{0,2,0}, x_{1,2,0}, x_{2,2,0}]_r$$

and

$$[x_{1,1,0}, x_{2,1,0}, x_{3,1,0}]_r.$$

Because of convexity of metric in a space of nonpositive curvature the function

$$d_{2,1}(t) = |x_{t,2,0} x_{t+1,1,0}|,$$

where $x_{t,j,0} = c_j(t)$, $c_j : [0, 2r] \rightarrow X$, $j = \overline{1, 2}$ — geodesic segments connecting $x_{0,j,0}$ with $x_{1,j,0}$, is a convex function. But

$$d_{2,1}(0) = d_{2,1}(1) = d_{2,1}(2) = 1,$$

hence $d_{2,1}(t) = 1$ for all $t \in [0, 2]$, and points (2.7) belong to a flat parallelogram

$$(2.8) \quad x_{0,2,0} x_{1,1,0} x_{3,1,0} x_{2,2,0}$$

isometrically embedded into X .

Similarly six points

$$x_{2,1,0}, \quad x_{3,1,0}, \quad x_{2,2,0}, \quad x_{1,3,0}, \quad x_{0,3,0}, \quad x_{1,2,0}$$

belong to isometrically embedded into X flat parallelogram

$$(2.9) \quad x_{2,1,0} x_{3,1,0} x_{1,3,0} x_{0,3,0}.$$

Parallelograms (2.8) and (2.9) have the common part — parallelogram

$$x_{2,1,0} x_{3,1,0} x_{2,2,0} x_{1,2,0},$$

and hence their union is nonconvex flat hexagon

$$x_{0,2,0} x_{1,1,0} x_{3,1,0} x_{1,3,0} x_{0,3,0} x_{1,2,0}.$$

It is isometric to a nonconvex hexagon in Euclidean plane, constructed by two equal parallelograms. Continuing so on we get a flat polygon

$$(2.10) \quad \begin{aligned} P = & x_{0,2,0} x_{1,1,0} x_{3,1,0} x_{2,2,0} x_{3,2,0} \dots \\ & \dots x_{3,p-1,0} x_{2,p,0} x_{3,p,0} x_{2,1,2p-1} x_{0,1,2p-1} x_{0,p,0} x_{1,p-1,0} \dots \\ & \dots x_{1,3,0} x_{0,3,0} x_{1,2,0}, \end{aligned}$$

(see Figure 1) which is isometric to a $(4p-2)$ -gon constructed from $2p-3$ parallelograms, consecutively intersecting each other.

P contains whole geodesic segments $x_{1,1,0} x_{1,1,2p-1}$ and $x_{2,1,0} x_{2,1,2p-1}$. Hence intersection of P with a flat strip F contains parallelogram

$$x_{1,1,0} x_{2,1,0} x_{2,1,2p-1} x_{1,1,2p-1}.$$

Furthermore P is contained in parallelogram $x_{0,1,0} x_{3,1,0} x_{3,1,2p-1} x_{0,1,2p-1}$ and hence in F . Since there are points of all r -sequences (2.4) in P , we may say that the whole tape is contained in F . \square

The width of p -tape (2.5) in X is a distance between parallel geodesics including r -sequences $\{x_{0,j,z}\}_{z \in \mathbb{Z}}$ and $\{x_{3,j,z}\}_{z \in \mathbb{Z}}$.

Lemma 2.11. *The width $s(p)$ of p -tape equals to*

$$s(p) = \frac{3\sqrt{4p-1}}{2p}.$$

Proof. Since every flat strip in $CAT(0)$ -space is isometric to a strip in Euclidean plane, it suffices to calculate a width of standard Euclidean p -tape. \square

Corollary 2.12. *For every flat strip containing a geodesic c , there exists a number $P > 0$, such that for all $p > P$ the strip contains a p -tape generated by a family of r -sequences of type (2.4) with $\{x_{0,j,z}\} \subset c$.*

Lemma 2.13. *Let the geodesic $c : (-\infty, +\infty) \rightarrow X$ contains r -sequence $\{x_z\}_{z \in \mathbb{Z}}$, $x_0 = c(0)$ of higher rank. Then for every rational $q := \frac{m}{n}$ and $p := kn$, multiple of n , every p -tape for which $x_{0,1,z} = x_z$ contains a point $c(q) = x_{0,j,z'}$ with*

$$j := n + 1 - km'$$

and

$$z' := q' + 1 - 2p + 2k(m' + 1),$$

where $q' := \left[q - \frac{1}{n} \right] - \text{integral}$ and $\frac{m'}{n} := \left\{ q - \frac{1}{n} \right\} - \text{fractional parts of the number } q - \frac{1}{n}$.

Proof. By direct calculation for a standard flat p -tape. \square

This yields, that for all p multiple to n every p -tape constructed with base points x_z of given r -sequence $\{x_z\}_{z \in \mathbb{Z}}$, contains points $c(\frac{m}{n})$ as elements with defined by p multiindex. We are ready now to prove the part of Theorem 0.4 in the case we consider.

Theorem 2.14. *Let metric space (X, d) and trial metric d' on X be as in Theorem 0.4. Assume that $c : (-\infty, +\infty) \rightarrow X$ is a geodesic in metric d bounding flat strip in X . Then c is geodesic bounding flat strip in metric d' and for all $t_1, t_2 \in \mathbb{R}$*

$$(2.15) \quad d'(c(t_1), c(t_2)) = d(c(t_1), c(t_2)).$$

Proof. We may assume that $t_1 = 0$. Let $\{x_z\}_{z \in \mathbb{Z}}$ be r -sequence lying in c such that $x_z = c(z)$. It is r -sequence of higher rank relatively both metrics d and d' and every p -tape in metric d based on $\{x_z\}_{z \in \mathbb{Z}}$ is p -tape in metric d' as well. When t is rational, equality (2.15) is a consequence of Lemma 2.13.

Let the number t be irrational and $t_n, t'_n \rightarrow t$ be a pair of sequences of its rational approximations from below and from above correspondingly. Without loss of generality we may assume that $t > 0$. Then

$$c(t) = \left(\bigcap_{n=1}^{\infty} B(c((t'_n - 1)), 1) \right) \cap \left(\bigcap_{n=1}^{\infty} B(c((t_n + 1)), 1) \right),$$

and for metric $D \in \{d, d'\}$ we have

$$t_n < D(c(0), c(t)) < t'_n,$$

implying the claim. \square

Remark 2.16. In other words, Theorem 2.14 asserts that the diagonal tube V allows to recover the metric of geodesic c bounding a flat strip.

Corollary 2.17. *V recovers a metric of any geodesic which virtually bounds a flat strip in X .*

Proof. The claim is a partial case of lemma 1.10, since every geodesic in metric d does not bounding flat strip is also geodesic in trial metric d' . \square

3. GEODESICS OF RANK ONE

3.1. Scissors.

Definition 3.1. We say that four complete geodesics $a, b, c, d : (-\infty, +\infty) \rightarrow X$ in $CAT(0)$ -space X form *scissors centered* in $x \in X$ if:

- $a(-\infty) = b(-\infty)$;
- $a(+\infty) = c(+\infty)$;
- $c(-\infty) = d(-\infty)$;
- $b(+\infty) = d(+\infty)$; and
- $b \cap c = x$.

Such configuration will be denoted as $\langle a, b, c, d; x \rangle$ (figure 2). Geodesics a and d above are called *bases* of scissors. One of them, for example a , will be labeled as *lowest base*. Writing $\langle a, b, c, d; x \rangle$ we put the lowest base first. One or both of bases may in general pass throw x_0 , but when X has a property of nonbranching of geodesics, such a situation is excluded.

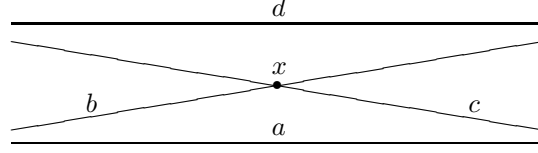
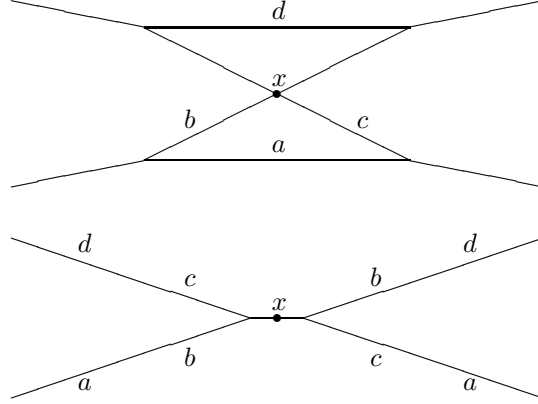
FIGURE 2. Scissors $\langle a, b, c, d; x \rangle$ 

FIGURE 3. Examples of scissors with branching of lines

Note that when scissors are given, they are not uniquely determined by a choice of their lowest base a and a center x in general: we must take into account the possibility of branching of geodesics b and c . Highest bases d_1 and d_2 of scissors $\langle a, b, c, d_1, x \rangle$ and $\langle a, b, c, d_2, x \rangle$ with common lines a, b, c may be parallel. Lines forming scissors may be partially or even totally attached. Some examples of scissors with branching of their lines are presented at figure 3.

The main advantage of using scissors consists in a scissors translation T , along their lowest base. It may be described as following. Assume that R_{ac} is metric transfer from geodesic line a to the line c generated by Busemann function $\beta_{a(+\infty)}$ as in subsection 1.5: any point $m \in a$ moves to the unique point $m' = R_{ac}(m) \in c$ with $\beta_{a(+\infty)}(m') = \beta_{a(+\infty)}(m)$. Similarly maps R_{cd} , R_{db} and R_{ba} are defined as metric transfers of corresponding lines. All maps above do not depend on the choice of Busemann functions β in classes defined by corresponding ideal points.

Definition 3.2. *Scissors translation* T is a composition

$$T := R_{ba} \circ R_{db} \circ R_{cd} \circ R_{ac} : a \rightarrow a .$$

Since all maps forming the translation T above are isometric maps, T is an isometry of a geodesic a preserving its direction. The displacement of the translation T i.e. difference $\beta_{a(-\infty)}(T(m)) - \beta_{a(-\infty)}(m)$ does not depend on the choice of the point $m \in a$. We denote this value as δT .

The term δT admits following approach. Let $\beta_{a-}, \beta_{a+}, \beta_{d-}$ and β_{d+} be four Busemann functions such that there exists points $p \in a$ and $q \in d$, satisfying equalities $\beta_{a-}(p) = \beta_{a+}(p) = 0$ and $\beta_{d-}(q) = \beta_{d+}(q) = 0$.

Theorem 3.3.

$$(3.4) \quad \delta T = \beta_{a-}(x) + \beta_{a+}(x) + \beta_{d-}(x) + \beta_{d+}(x) \geq 0,$$

where x is a center of scissors $\langle a, b, c, d; x \rangle$ with translation T . Moreover, if none of geodesics a or d bounds flat strip and if $a \cap d = \emptyset$, then $\delta T > 0$

Proof. First, note that sums $\beta_{a-}(x) + \beta_{a+}(x)$ and $\beta_{d-}(x) + \beta_{d+}(x)$ are independent on the choice of points $p \in a$ and $q \in d$, since, for example, substituting point p by $p' \in a$, one adds constants $\beta_{a-}(p')$ and $\beta_{a+}(p')$, where $\beta_{a-}(p') = -\beta_{a+}(p')$, to functions β_{a-} and β_{a+} correspondingly. We have for δT :

$$\delta T = t' - t,$$

where

$$a(t') = T(a(t)).$$

If $d(s) = R_{cd} \circ R_{ac}(a(0))$, then for all $t \in \mathbb{R}$

$$d(s+t) = R_{cd} \circ R_{ac}(a(t)).$$

Analogously, if $a(t) = R_{ba} \circ R_{db}(d(0))$, then for all $s \in \mathbb{R}$

$$a(t+s) = R_{ba} \circ R_{db}(d(s)).$$

Set $p := a(0) = R_{ac}^{-1}(x)$ and $q := d(0) = R_{db}^{-1}(x)$. Then

$$\beta_{a+}(x) = \beta_{a+}(p) = 0,$$

$$\beta_{d+}(x) = \beta_{d+}(q) = 0$$

and

$$T(p) = R_{ba} \circ R_{db} \circ R_{cd}(x) = R_{ba} \circ R_{db}(d(s)),$$

where $s = \beta_{d-}(x)$. We have:

$$T(p) = a(\beta_{d-}(x) + t),$$

with $t := \beta_{a-}(x)$. Hence displacement of a point $p \in a$, and consequently of any point of a equals to

$$\delta T = \beta_{a-}(x) + \beta_{d-}(x) - 0 = \beta_{a-}(x) + \beta_{a+}(x) + \beta_{d-}(x) + \beta_{d+}(x).$$

We have $\delta T \geq 0$, since x lies in intersections $hb(a(+\infty), x) \cap hb(a(-\infty), x)$ and $hb(d(+\infty), x) \cap hb(d(-\infty), x)$ of horoballs. When additional conditions

are assumed, we may suppose without loss of generality, that $x \notin a$. In this case $\beta_{d-}(x) + \beta_{d+}(x) \geq 0$ and $\beta_{a-}(x) + \beta_{a+}(x) > 0$. \square

3.2. Shadows.

Definition 3.5. Fix a point x_0 . The complete shadow of a point x_0 with respect to point $y \in \overline{X} \setminus \{x_0\}$ is by definition a set

$$\text{Shadow}_y(x_0) := \{z \in \overline{X} \mid \exists [yz] \quad x_0 \in [yz]\}.$$

Assuming of existence is necessary here only if both points y, z are infinite: $y, z \in \partial_\infty X$. Spherical shadow of a point x_0 of radius $\rho > 0$ relatively point $y \in \overline{X}$ is intersection $\text{Shadow}_y(x_0, \rho)$ of its complete shadow $\text{Shadow}_y(x_0)$ with the sphere $S(x, \rho)$. In particular, when $\rho = +\infty$

$$\text{Shadow}_y(x_0, +\infty) := \partial_\infty(\text{Shadow}_y(x_0)) := \text{Shadow}_y(x_0) \cap \partial_\infty X.$$

The horosphere $\mathcal{HS}_{y,z}$ where $z \in X$ is such a point that $\beta_y(z) - \beta_y(x_0) = \rho$, will be denoted as $\mathcal{HS}_{y,\rho}$. Here β_y is arbitrary Busemann function centered in $y \in \partial_\infty X$. Following properties of shadows are obvious.

- (1) Sets $\text{Shadow}_y(x_0) \cup \{x_0\}$ and $\text{Shadow}_y(x_0, \rho)$ are closed in \overline{X} for all $\rho > 0$;
- (2) $\text{Shadow}_y(x_0) = \overline{\bigcup_{\rho > 0} \text{Shadow}_y(x_0, \rho)} \setminus \{x_0\}$;
- (3) If $y \in X$, then $\text{Shadow}_y(x_0, \rho) = S(y, |xy| + \rho) \cap S(x, \rho)$ for all $\rho > 0$;
- (4) If $y \in \partial_\infty X$, then $\text{Shadow}_y(x_0, \rho) = (\mathcal{HS}_{y,\rho}) \cap S(x, \rho)$ for all $\rho > 0$;
- (5) If $\angle_{x_0}(y, z) = 0$, then $\text{Shadow}_y(x_0) = \text{Shadow}_z(x_0)$.

The statement 5 implies consequence:

Corollary 3.6. If the direction of the ray $c: [0, \infty) \rightarrow X$ in a point x_0 with $c(0) = x_0$ and $c(|xy|) = y$, has unique inverse in x_0 , then for any two $z', z'' \in \text{Shadow}_y(x_0)$

$$\text{Shadow}_{z'}(x_0) = \text{Shadow}_{z''}(x_0).$$

For sufficiently small $\varepsilon > 0$ $\mathcal{N}_\varepsilon(\text{Shadow}_y(x_0, \rho))$ will denote ε -neighbourhood of spherical shadow $\text{Shadow}_y(x_0, \rho)$

- in the sphere $S(y, |xy| + \rho)$, if $y \in X$, or
- in the horosphere $\mathcal{HS}_{y,\rho}$, if $y \in \partial_\infty X$.

Theorem 3.7. For all $x_0 \in X, y \in \overline{X} \setminus \{x_0\}, 0 < \rho < +\infty$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1 \in B(x_0, \delta)$, satisfying equality $|yx_1| = |yx_0|$ (or $\beta_y(x_1) = \beta_y(x_0)$ when $y \in \partial_\infty X$), inclusion

$$\text{Shadow}_y(x_1, \rho) \subset \mathcal{N}_\varepsilon(\text{Shadow}_y(x_0, \rho))$$

holds.

Proof. We prove the statement in the case $y \in X$. Situation when $y \in \partial_\infty X$ is similar. Assume that for any $\delta > 0$ there exists a point $x_\delta \in B(x_0, \delta) \cap S(y, |yx_0|)$ and a point $z_\delta \in S(y, |yx_0| + \rho) \setminus \mathcal{N}_\varepsilon(\text{Shadow}_y(x_0, \rho))$, for which $x_\delta \in [yz_\delta]$. We take a sequence $\delta_n \rightarrow 0$ and corresponding sequences of points x_{δ_n} and z_{δ_n} . Then $x_{\delta_n} \rightarrow x_0$. Since the space X is finitely compact, one may choose converging subsequence from z_{δ_n} . We may assume that sequence z_{δ_n} converges to a point $z \in S(y, |yx_0| + \rho)$ itself. Parametrizing the segment $\gamma = [yz]$ by arclength, we obtain

$$(3.8) \quad |x_0 \gamma(|yx_0|)| \leq |x_0 x_{\delta_n}| + |x_{\delta_n} \gamma(|yx_0|)| \leq \delta_n + |z_{\delta_n} z|.$$

The value on the right hand of (3.8) vanishes when $n \rightarrow \infty$, hence the constant on the left hand is 0. Consequently, the point $z \in \text{Shadow}_y(x_0, \rho)$, and points z_n belong to $\mathcal{N}_\varepsilon(\text{Shadow}_y(x_0, \rho))$ when n is sufficiently large. Contradiction. \square

Let a set $\mathcal{V} \subset \partial_\infty X$ and a number $K, \varepsilon > 0$ be given. (y, K, ε) -neighbourhood of \mathcal{V} is by definition a set

$$\mathcal{N}_{y, K, \varepsilon}(\mathcal{V}) := \{\zeta \in \partial_\infty X \mid \exists \xi \in \mathcal{V} \quad \zeta \in \mathcal{U}(\xi, y, K, \varepsilon)\},$$

where

$$\mathcal{U}(\xi, x_0, K, \delta) := \{\eta \in \partial_\infty X \mid |c(K)d(K)| < \varepsilon, \quad c = [y, \xi]; d = [y, \eta]\}.$$

Following claim in fact is only reformulated statement of the theorem 3.7 for a case $y \in X$.

Corollary 3.9. *For any point $y \in X$ and any (y, K, ε) -neighbourhood $\mathcal{N}_{y, K, \varepsilon}(\partial_\infty(\text{Shadow}_y(x_0)))$ of shadow at infinity $\partial_\infty(\text{Shadow}_y(x_0))$ there exists $\delta > 0$, such that for any $x_1 \in B(x_0, \delta)$, with $|yx_1| = |yx_0|$, we have*

$$\partial_\infty(\text{Shadow}_y(x_1)) \subset \mathcal{N}_{y, K, \varepsilon}(\partial_\infty(\text{Shadow}_y(x_0))).$$

3.3. The geometry of ideal boundary of $CAT(0)$ -space. In this subsection we recall some well-known facts from asymptotic geometry of Hadamard spaces. We refer the reader to [BH] for more detailed considerations.

First of all, we note that given a point $y_0 \in X$, an ideal point $\xi \in \partial_\infty X$ and a number $t > 0$, a family of sets

$$(3.10) \quad \mathcal{B}_{y_0, \xi} := \{\mathcal{U}_{\delta, t}(y_0, \xi) \mid \delta, t > 0\}$$

form a basis of neighbourhoods of point ξ in the cone topology in $\partial_\infty X$. Here

$$\mathcal{U}_{\delta, t}(y_0, \xi) := \{\eta \in \partial_\infty X \mid |c(t)d(t)| < \delta\},$$

and $c, d : [0, +\infty) \rightarrow X$ are rays emanating from y_0 in $\xi, \eta \in \partial_\infty X$ directions correspondingly.

The space X instead of the cone topology on ideal boundary $\partial_\infty X$ has one induced by so called *angle metric*. For $\xi, \eta \in \partial_\infty X$ angle distance is by definition equal to

$$\angle(\xi, \eta) := \sup\{\angle_x(\xi, \eta) \mid x \in X\}.$$

The interior metric on $\partial_\infty X$ associated with angle metric is called *Tits metric* and denoted as Td . The two metrics on $\partial_\infty X$ are equivalent in the sense that they induce the same topology on $\partial_\infty X$. We will write $\partial_T X$ for denoting the ideal boundary equipped with Tits metric. We need following two propositions.

Proposition 3.11 ([BH], Proposition 9.5). *Angle metric considered as a function $(\xi, \eta) \rightarrow \angle(\xi, \eta)$ is lower semicontinuous with respect to cone topology: for all $\varepsilon > 0$ there exist neighbourhoods \mathcal{U} of point ξ and \mathcal{V} of point η , such that for all $\xi' \in \mathcal{U}$ $\eta' \in \mathcal{V}$ inequality*

$$\angle(\xi', \eta') > \angle(\xi, \eta) - \varepsilon$$

holds.

As a corollary, the Tits metric is also lower semicontinuous in the cone topology.

Proposition 3.12 ([BH], Proposition 9.21). *Let X be a proper $CAT(0)$ -space and let ξ_0 and ξ_1 be distinct points of $\partial_\infty X$.*

- (1) *If $Td(\xi_0, \xi_1) > \pi$ then there is a geodesic $c : \mathbb{R} \rightarrow X$ with $c(+\infty) = \xi_0$ and $c(-\infty) = \xi_1$,*
- (2) *If there is no geodesic $c : \mathbb{R} \rightarrow X$ with $c(+\infty) = \xi_0$ and $c(-\infty) = \xi_1$, then $Td(\xi_0, \xi_1) = \angle(\xi_0, \xi_1)$ and there is a geodesic segment in $\partial_T X$ joining ξ_0 and ξ_1 ,*
- (3) *If $c : \mathbb{R} \rightarrow X$ is a geodesic, then $Td(c(-\infty), c(+\infty)) \geq \pi$ with equality iff c bounds a flat half-plane,*
- (4) *If the diameter of the Tits boundary $\partial_T X$ is π , then every geodesic line in X bounds a flat half-plane.*

3.4. Points with uniqueness of inverse direction. Directions $\xi, \eta \in \Sigma_x X$ are called *mutually inverse* if $\angle_x(\xi, \eta) = \pi$. In the case of geodesically complete $CAT(0)$ -space two directions $\xi, \eta \in \Sigma_x X$ are mutually inverse iff there exists a geodesic throw x whose positive direction in x is ξ and negative one is η .

Given geodesic a in $CAT(0)$ -space X we denote $\omega^+(a)$ a set of points $x \in a$ with the property that the positive direction $\xi \in \Sigma_x X : [xa(+\infty)] \in \xi$ of a has more than one inverse direction. Similarly $\omega^-(a)$ be the set of points where the negative direction $\eta \in \Sigma_x X : [xa(-\infty)] \in \eta$ of a has more than one inverse.

Theorem 3.13. *Let a be a geodesic in a geodesically complete locally compact $CAT(0)$ -space X . Then sets $\omega^+(a)$ and $\omega^-(a)$ are at most countable.*

Proof. For $\phi > 0$ consider set $\Omega_\phi^+(a) \subset \omega^+(a)$ defined as following. $x \in \Omega_\phi^+(a)$ iff there exists a direction $\zeta \in \Sigma_x X$ inverse to direction of the ray $[xa(+\infty)]$, such that $\angle_x(\zeta, a(-\infty)) > \phi$. We will show that intersection of $\Omega_\phi^+(a)$ with any segment $[xy] \subset a$ is finite.

Really, assume that there exists a segment $[xy] \subset a$ containing infinite sequence $\{a(t_n)\}_{n=1}^\infty \subset \Omega_\phi^+(a)$. We may assume that $x = a(0)$ and $y = a(-L)$, where $L = |xy|$. For a point $a(t_n)$ we set $z_n \in S(x, 2L)$ a point such that $a(t_n) \in [xz_n]$ and $\angle_{a(t_n)}(a(-\infty), z_n) > \phi$. Then for $n \neq k$ we have $|z_n z_k| > 2L \sin \frac{\phi}{2}$ and the sequence $\{z_n\}_{n=1}^\infty$ does not contain any fundamental subsequence, contradicting finitely compactness of X .

Because of $\omega^+(a) = \bigcup_{\phi > 0} \Omega_\phi^+(a)$ and $\Omega_\phi^+(a) \subset \Omega_\psi^+(a)$ for $\psi < \phi$, we have the claim for $\omega^+(a)$. Consideration of $\omega^-(a)$ is similar. □

Remark 3.14. Otsu and Tanoue in [OT] introduce the following notion. For $\delta > 0$ and $y \in X$ point x is called δ -*branched point* of y if the diameter of the set $\{v \in \Sigma_x X \mid \angle_x(v_{xy}v) = \pi\}$ is not smaller then δ . Here $v_{xy} \in \Sigma_x X$ is the direction of the segment $[xy]$. Such a notion has evident extension on the case $y \in \bar{X}$. In fact, every set $\Omega_\phi^+(a)$ is contained in a set of ϕ -branching points of $a(+\infty)$ lying in a , which also has finite intersection with every segment $[xy] \subset a$.

Example 3.15. Every point of a geodesic $a \subset X$ may occur to be a branching point. For example, exclude from Euclidean plane the interior domain bounded by a parabola and attach a flat half-plane on its place. We get a $CAT(0)$ -space X . The parabola a dividing two flat domains is its geodesic and every its point is a point of branching of geodesics in both directions. However we have $\omega^+(a) = \omega^-(a) = \emptyset$, since angle between any two branches of a in its opposite direction starting from the same point vanishes.

Example 3.16. Sets $\omega^+(a)$ and $\omega^-(a)$ may occur to be dense in a . To see this one may take a convex continuous natural parametrised curve on which every point has a positive and negative semitangents, not opposite to each other in rational points, instead of the parabola in previous example.

3.5. Existence of scissors. We prove the existence theorem for scissors in this subsection.

Theorem 3.17. *Let $a : (-\infty, +\infty) \rightarrow X$ be a geodesic of strictly rank one. Let $x_0 \in a \setminus (\omega^+(a) \cup \omega^-(a))$ be a point where both directions of a have unique inverse. Then there exists a geodesic a' with $a'(0) = x_0$ and $\angle_{x_0}(a(+\infty), a'(+\infty)) = 0$ with following property. For every neighbourhood \mathcal{U} of a triple*

$$(a'(+\infty), a'(-\infty), x_0) \in \partial_\infty X \times \partial_\infty X \times X$$

there exists a triple $(\xi, \eta, x) \in \mathcal{U}$ with $x \neq x_0$ and geodesics $b' = [a'(-\infty)\xi]$, $c' = [\eta a'(+\infty)]$ and $d' = [\eta\xi]$, forming scissors $\langle a', b', c', d'; x \rangle$.

Proof. We note that by the condition on the rank of a , any geodesic a' with $a'(+\infty) \in \partial_\infty(\text{Shadow}_{a(-\infty)}(x_0))$ or $a'(-\infty) \in \partial_\infty(\text{Shadow}_{a(+\infty)}(x_0))$ has rank 1. First we show that there exists scissors with lowest base a and center x arbitrary closed to x_0 . In view of the first remark, the considerations will be applicable also for a geodesic a' passing throw x_0 in the same direction.

Take points $y' = a(-\rho)$ and $y'' = a(\rho)$, where $\rho > 0$. We have

$$\partial_\infty(\text{Shadow}_{a(-\infty)}(x_0)) = \partial_\infty(\text{Shadow}_{y'}(x_0))$$

and

$$\partial_\infty(\text{Shadow}_{a(+\infty)}(x_0)) = \partial_\infty(\text{Shadow}_{y''}(x_0)).$$

Low semicontinuous function

$$Td : \partial_\infty X \times \partial_\infty X \rightarrow \mathbb{R}_+ \cup \{+\infty\}$$

attains its minimum on the compact set

$$Q = \partial_\infty(\text{Shadow}_{y'}(x_0)) \times \partial_\infty(\text{Shadow}_{y''}(x_0)),$$

and inequality

$$(3.18) \quad \min(Td)|_Q > \pi$$

holds as a consequence of the condition on the rank of a . Moreover, there exists neighbourhoods

$$\mathcal{N}' := \mathcal{N}_{y', K, \varepsilon}(\partial_\infty(\text{Shadow}_{y'}(x_0)))$$

and

$$\mathcal{N}'' := \mathcal{N}_{y'', K, \varepsilon}(\partial_\infty(\text{Shadow}_{y''}(x_0)))$$

with some $K > \rho$ and $\varepsilon > 0$, for which

$$(3.19) \quad \inf\{Td(\xi, \eta) \mid (\xi, \eta) \in \mathcal{N}' \times \mathcal{N}''\} > \pi.$$

Choose δ_1 -neighbourhood $B(x_0, \delta_1)$ of a point x_0 defined by Corollary 3.9 relatively $\mathcal{N}_{y', K, \varepsilon/2}(\partial(\text{Shadow}_{y'}(x_0, \rho)))$ and $\mathcal{N}_{y'', K, \varepsilon/2}(\partial(\text{Shadow}_{y''}(x_0, \rho)))$.

Let also $\mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(-\infty)}(x_0, \rho)))$ and $\mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(+\infty)}(x_0, \rho)))$ be $\varepsilon/2$ -neighbourhoods of boundaries of shadows of point x_0 relatively points $a(-\infty)$ and $a(+\infty)$.

From Theorem 3.7 there exists δ_2 -neighbourhood $B(x_0, \delta_2)$ of point x_0 , such that for every $x' \in B(x_0, \delta_2)$ inclusions

$$\text{Shadow}_{a(-\infty)}(x', \rho) \subset \mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(-\infty)}(x_0, \rho)))$$

and

$$\text{Shadow}_{a(+\infty)}(x', \rho) \subset \mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(+\infty)}(x_0, \rho)))$$

hold.

Set $\delta_0 := \min\{\delta_1, \delta_2\}$. Then for any point $x \in \mathcal{U}_{\delta_0}(x_0)$ and geodesics b and c satisfying conditions

- $b(0) = c(0) = x$,
- $b(-\infty) = a(-\infty)$ and
- $c(+\infty) = a(+\infty)$

we get

$$(3.20) \quad b(\rho) \in \mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(-\infty)}(x_0, \rho)))$$

and

$$c(-\rho) \in \mathcal{N}_{\varepsilon/2}(\partial(\text{Shadow}_{a(+\infty)}(x_0, \rho)))$$

We will show that

$$(3.21) \quad b(+\infty) \in \mathcal{N}'$$

and

$$(3.22) \quad c(-\infty) \in \mathcal{N}''.$$

Given geodesic ray $\gamma = [y'b(+\infty)]$ with length parametrisation

$$\gamma : [0, +\infty) \rightarrow X$$

and geodesic line a' passing throw x_0 such that $a'(+\infty) \in \partial_\infty(\text{Shadow}_{y'}(x_0))$ we have

$$|\gamma(2\rho)a'(\rho)| \leq |\gamma(2\rho)b(\rho)| + |b(\rho)a'(\rho)|.$$

First item has estimation

$$|\gamma(2\rho)b(\rho)| \leq |\gamma(0)b(-\rho)| = |a(-\rho)b(-\rho)| \leq |a(0)b(0)| < \frac{\varepsilon}{2}.$$

Because of (3.20) one may choose the geodesic a' such that second item satisfies to inequality

$$|b(\rho)a'(\rho)| < \frac{\varepsilon}{2}.$$

Finally we get

$$|\gamma(2\rho)a'(\rho)| < \varepsilon,$$

proving the inclusion (3.21). Inclusion (3.22) is similar.

So because of (3.19) there exists a geodesic d in X connecting points $c(-\infty)$ and $b(+\infty)$ such that we have scissors $\langle a, b, c, d; x \rangle$.

Now we take a sequences $\delta_n \rightarrow 0$ of scales and $\langle a_n, b_n, c_n, d_n; x_n \rangle$ of corresponding scissors for which $|x_0 x_n| < \delta_n$. Choose as $\xi, \eta \in \partial_\infty X$ limit points of sequences $b_n(+\infty)$ and $c_n(-\infty)$ correspondingly. Then $\xi \in \partial_\infty(\text{Shadow}_{a(-\infty)}(x_0))$ and $\eta \in \partial_\infty(\text{Shadow}_{a(+\infty)}(x_0))$. The points above may be connected by a geodesic $a' = [\eta\xi]$ in X such that $a'(0) = x_0$ (cf. inequality (3.18)). For any $\delta > 0$ there exists scissors $\langle a', b', c', d'; x \rangle$ with lowest base a' and $|xx_0| < \delta$. Note that the point x may always be different of x_0 and not belong to a' .

It is left to show that such scissors may be chosen with $b'(+\infty)$ arbitrary closed to ξ , and $c'(-\infty)$ arbitrary closed to η in the sence of cone topology on $\partial_\infty X$. It may be done with applying the same method as above but

with neighborhoods of points $a'(\pm\infty)$ instead of neighborhoods of shadows at infinity of point x_0 . Construction of a line a' guarantees that sets

$$C_{-\infty}(B(a'(K), \varepsilon)) = \bigcup_{y \in B(a'(K), \varepsilon)} [a'(-\infty)y]$$

and

$$C_{+\infty}(B(a'(-K), \varepsilon)) = \bigcup_{z \in B(a'(-K), \varepsilon)} [za'(+\infty)]$$

has nonempty intersection for any $\varepsilon, K > 0$, and moreover

$$\begin{aligned} & C_{-\infty}(\mathcal{N}_\varepsilon(a'(K))) \cap C_{+\infty}(\mathcal{N}_\varepsilon(a'(-K))) \\ & \cap (X \setminus (\text{Shadow}_{y'}(x_0) \cup \text{Shadow}_{y''}(x_0) \cup \{x_0\}) \cap B(x_0, \varepsilon)) \neq \emptyset. \end{aligned}$$

This fact provides for necessary construction. \square

Remark 3.23. Note that geodesics a and a' are connected by the asymptotic chain $a = a_0, a_1, a_2 = a'$, where the geodesic a_1 is obtained as union of rays $[x_0a(-\infty)]$ and $[x_0a'(+\infty)]$.

3.6. Continuity of the displacement function. The goal of this subsection is a theorem 3.29 which is a theorem of continuity for a displacement function δ defined by the scissors translations along given geodesic a as a function defined on appropriate subset of $\partial_\infty X \times \partial_\infty X \times X$. First we need some estimation of distance between projections.

We say that a point $x \in X$ *projects* to a point $x_0 \in a$, if x_0 is nearest to x point of a geodesic a . It is called a *projection* of x . Every point $x \in X$ has unique projection on any given geodesic a .

Theorem 3.24. *Let $d : (-\infty, +\infty) \rightarrow X$ be a strongly rank one geodesic, $x \notin d$ be a point in X and x_1 — its projection onto d . Then for any $\varepsilon > 0$ points $\xi = d(+\infty)$ and $\eta = d(-\infty) \in \partial_\infty X$ have neighbourhoods \mathcal{U}_+ and \mathcal{U}_- correspondingly, such that if a geodesic d' connects points $d'(+\infty) \in \mathcal{U}_+$ and $d'(-\infty) \in \mathcal{U}_-$, and $x'_1 \in d'$ is a projection of the point x onto d' , then $|x_1x'_1| < \varepsilon$.*

The first step of its proof is the next simple lemma.

Lemma 3.25. *Let d be a geodesic in the $CAT(0)$ -space X and x' be the projection of the point x onto d . Then for any point $y \in d$*

$$|x'y| \leq \sqrt{|xy|^2 - |xx'|^2}.$$

Proof. Assume that $|x'y| > \sqrt{|xy|^2 - |xx'|^2}$. Consider a comparison triangle $\overline{xx'y}$ of a triangle $xx'y$. We have by assumption $\angle_{\overline{x}}(\overline{x}, \overline{y}) < \pi/2$. Hence there exists a point $\overline{m} \in \overline{xy}$, with $|\overline{x}\overline{m}| < |\overline{x}\overline{x'}|$. Its corresponding point m of triangle $xx'y$ satisfies inequality $|xm| \leq |\overline{x}\overline{m}| < |xx'|$. The contradiction proves the claim. \square

Proof of the theorem 3.24. Let $d : (-\infty, +\infty) \rightarrow X$ be given geodesic and $\sigma > 0$ an arbitrary number. We denote $HS_\xi(t, \rho)$ the intersection of a horosphere $\mathcal{HS}_{\xi, d(t)}$ centered at a point $\xi \in \partial_\infty X$ with $B(d(t), \rho)$.

We need the fact that the cone topology of $CAT(0)$ -space X has a base consisting of sets

$$\mathcal{U}(\xi, x_0, K, \delta) := \{\eta \in \partial_\infty X \mid |c(K)c_*(K)| < \delta \quad c = [x_0, \xi]; c_* = [x_0, \eta]\}$$

Choose points $x_{-K} = d(-K)$ and $x_K = d(K)$, where $K > 0$ is a sufficiently large number. In particular one may assume for K to satisfy the following condition: if $y \in B(x_{-K}, \sigma)$, then $[yd(+\infty)] \cap HS_{d(+\infty)}(K, \sigma/4) \neq \emptyset$, and if $z \in B(x_K, \sigma)$, then $[zd(-\infty)] \cap HS_{d(-\infty)}(-K, \sigma/4) \neq \emptyset$. This condition can be accomplished because d has strictly rank one.

Denote as

$$\mathcal{U}_{+\infty}(K, \sigma) := \left\{ \xi \in \partial_\infty X \mid \forall y \in B(x_{-K}, \sigma) \quad [y\xi] \cap HS_{d(+\infty)}\left(K, \frac{\sigma}{2}\right) \neq \emptyset \right\}$$

and

$$\mathcal{U}_{-\infty}(K, \sigma) := \left\{ \eta \in \partial_\infty X \mid \forall z \in B(x_K, \sigma) \quad [z\eta] \cap HS_{d(-\infty)}\left(-K, \frac{\sigma}{2}\right) \neq \emptyset \right\}$$

sets of those ideal points which serve as centers of projections onto corresponding horospheres, such that a ball $B(x_{-K}, \sigma)$ moves inside a ball $B(x_K, \sigma/2)$, and a ball $B(x_K, \sigma)$ moves inside a ball $B(x_{-K}, \sigma/2)$. Sets $\mathcal{U}_{\pm\infty}(K, \sigma)$ are nonempty, since $d(+\infty) \in \mathcal{U}_{+\infty}(K, \sigma)$ and $d(-\infty) \in \mathcal{U}_{-\infty}(K, \sigma)$. Furthermore, points $\gamma(\pm\infty)$ are interior points of sets $\mathcal{U}_{\pm\infty}(K, \sigma)$. This is a consequence of the definition of the cone topology on $\partial_\infty X$ and local compactness of X .

For an arbitrary point $x \notin d$ let x_1 be its projection onto d . Fix $\varepsilon > 0$. One may assume that $\varepsilon < \min\{|xx_1|, 1\}$, because it is sufficient to make all considerations for arbitrary small ε . Set

$$\sigma := \frac{\varepsilon^2}{9|xx_1|}.$$

For K as above we denote $\mathcal{U}_\pm \subset \partial_\infty X$ neighbourhoods of ideal points $d(\pm\infty)$, contained correspondingly in $\mathcal{U}_{\pm\infty}(K, \sigma)$.

We show that neighbourhoods \mathcal{U}_\pm are the ones satisfying the claim of the theorem: if the geodesic d' has ends $d'(\pm\infty) \in \mathcal{U}_\pm$ and $x'_1 \in d'$ is a projection of a point x onto d' , then $|xx'_1| < \varepsilon$. For this we show that $d' \cap \overline{HS}_{d(+\infty)}(K, \sigma/2) \neq \emptyset$ and $d' \cap \overline{HS}_{d(-\infty)}(-K, \sigma/2) \neq \emptyset$ at first.

Indeed, let $\pi_+ : HS_{d(-\infty)}(-K, \sigma) \rightarrow HS_{d(+\infty)}(K, \sigma/2)$ be the projection map centered in $\gamma'(+\infty)$ and $\pi_- : HS_{\gamma(+\infty)}(K, \sigma) \rightarrow HS_{\gamma(-\infty)}(-K, \sigma/2)$ be analogous projection centered in $d'(-\infty)$. Then by the condition on the rank of d' , the composition $\pi_- \circ \pi_+ : HS_{d(-\infty)}(-K, \sigma) \rightarrow HS_{d(-\infty)}(-K, \sigma/2)$ is a continuous map which is contraction operator on $HS_{d(-\infty)}(-K, \sigma)$. Since the closure $\overline{HS}_{d(-\infty)}(-K, \sigma/2)$ is compact, the contraction operator $\pi_- \circ \pi_+$

has unique fixed point in $\overline{HS}_{d(-\infty)}(-K, \sigma/2)$. Let such a point z have an image $y = \pi_+(z)$. Then $y \in [z d'(+\infty)]$ and $z \in [y d'(-\infty)]$. The union of rays $[z d'(+\infty)] \ni y$ and $[y d'(-\infty)] \ni z$ gives us precisely a geodesic d' , since it does not bound any flat strip.

It is only left to estimate the distance $|x'_1 x_1|$ to complete the proof. Since d' passes through interior points of balls $B(-K, \sigma)$ and $B(K, \sigma)$, there exists a point $m \in d'$, with $|m x_1| < \sigma$. Then we have:

$$(3.26) \quad |x'_1 x_1| \leq |x'_1 m| + |m x_1| \leq \sqrt{|x m|^2 - |x x'_1|^2} + \sigma.$$

By the triangle inequality we have for point m

$$(3.27) \quad |x m| < |x x_1| + \sigma,$$

and since x_1 is the nearest to x point of d and the distance of x'_1 to d is less than σ , the estimate

$$(3.28) \quad |x x'_1| > |x x_1| - \sigma$$

holds.

Substituting inequalities (3.27) and (3.28) to (3.26), we get

$$\begin{aligned} |x'_1 x_1| &< \sqrt{(|x x_1| + \sigma)^2 - (|x x_1| - \sigma)^2} + \sigma = 2\sqrt{\sigma |x x_1|} + \sigma \\ &= \sqrt{\sigma}(2\sqrt{|x x_1|} + \sqrt{\sigma}) < 3\sqrt{\sigma |x x_1|} = \varepsilon. \end{aligned}$$

□

Fix a strictly rank one geodesic $a : \mathbb{R} \rightarrow X$. Set $Z(a) \subset \partial_\infty X \times \partial_\infty X \times X$ to be a subset consisting of all triples $(\xi, \eta, x) \in \partial_\infty X \times \partial_\infty X \times X$ such that there exists scissors $\langle a, b, c, d; x \rangle$ with $b(+\infty) = \xi$ and $c(-\infty) = \eta$.

Theorem 3.29. *The displacement function δ is continuous on the set $Z(a)$.*

Proof. We use the equality (3.4). Fix a triple $(\xi_0, \eta_0, x_0) \in Z(a)$. It means that x_0 is the center of scissors $\langle a, b_0, c_0, d_0; x_0 \rangle$, where $b_0(+\infty) = \xi_0$ and $c_0(-\infty) = \eta_0$.

Fix an arbitrary $\varepsilon > 0$. Then first, by continuity of Busemann functions β_{a-} and β_{a+} there exists σ_1 such that if for a point $x' \in X$ inequality $|x_0 x'| < \sigma_1$ holds, then

$$(3.30) \quad |\beta_{a+}(x') + \beta_{a-}(x') - \beta_{a+}(x_0) - \beta_{a-}(x_0)| < \varepsilon/2.$$

Using the theorem 3.24 we choose neighbourhoods \mathcal{U}_+ and \mathcal{U}_- of points $d_0(+\infty) = \xi_0$ and $d_0(-\infty) = \eta_0 \in \partial_\infty X$ with the condition: for any geodesic d' such that $d'(-\infty) \in \mathcal{U}_-$ and $d'(+\infty) \in \mathcal{U}_+$, the projection x_1 of x_0 onto d_0 and projection x'_1 of x_0 onto d' satisfy inequality $|x_1 x'_1| < \varepsilon/4$. For these neighbourhoods there exists σ_2 such that if the point x' satisfy conditions

- a) there exists a ray $[x' \xi']$, inverse to the ray $[x' a(-\infty)]$ with $\xi' \in \mathcal{U}_+$,
- b) there exists a ray $[x' \eta']$, inverse to the ray $[x' a(+\infty)]$ with $\eta' \in \mathcal{U}_-$,
- c) $d' : (-\infty, +\infty) \rightarrow X$ is a geodesic connecting points ξ' and η' , and

d) $|x_0 x'| < \sigma_2$,
then

$$|\beta_{d'-}(x') - \beta_{d_0-}(x_0)| < \varepsilon/4$$

and

$$|\beta_{d'+}(x') - \beta_{d_0+}(x_0)| < \varepsilon/4.$$

Here $\beta_{d'-}$ and $\beta_{d'+}$ are Busemann functions corresponding to points $d'(-\infty)$ and $d'(+\infty)$ and satisfying the condition $\beta_{d'-}(x'_1) = \beta_{d'+}(x'_1) = 0$, and β_{d_0-} , β_{d_0+} — analogous Busemann functions for the geodesic d_0 . As a result, for $\sigma = \min\{\sigma_1, \sigma_2\}$ and for all x' such that $|x_0 x'| < \sigma$ and conditions a), b), c) above holds, we have an estimation:

$$(3.31) \quad |\beta_{d'-}(x') + \beta_{d'+}(x') - \beta_{d_0-}(x_0) - \beta_{d_0+}(x_0)| \leq \varepsilon/2.$$

Taking $\mathcal{N} = \mathcal{U}_+ \times \mathcal{U}_- \times B(x_0, \sigma)$ as neighbourhood of triple (ξ_0, η_0, x_0) and comparing inequalities (3.30) and (3.31) with representation (3.4), we get a condition of the continuity for the function δ as function of triple (ξ, η, x) in the point $(\xi_0, \eta_0, x_0) \in Z(a)$. \square

Theorem 3.32. *Let $x_0 \in a$ be an arbitrary point of a geodesic a of strictly rank one. Then $\delta(\xi, \eta, x) \rightarrow 0$ when $(\xi, \eta, x) = (b(+\infty), c(-\infty), x)$ tends to $(a(+\infty), a(-\infty), x_0)$ in the sense of topology in the set $Z(a)$ inherited from $\partial_\infty X \times \partial_\infty X \times X$.*

Proof. Let A be a set of lines a' which passes through x_0 such that $a'(+\infty) \in \text{Shadow}_{a(-\infty)}(x_0, +\infty)$ and $a'(-\infty) \in \text{Shadow}_{a(+\infty)}(x_0, +\infty)$. For $a' \in A$ we define "closed" scissors $\langle a', a', a', a'; x_0 \rangle$ as a system of four items of a line a' and a center x_0 . They also have a transformation map $T = \text{id}_{a'}$, for which $\delta T = 0$ and a function δ remains continuous when it is defined on $Z(a) \cup \{(a'(+\infty), a'(-\infty), x) | x \in a', a' \in A\}$ by equality $\delta(a'(+\infty), a'(-\infty), x) = 0$ for $x \in a'$. \square

Following evident corollary will be the key point in remainder of the proof.

Corollary 3.33. *Let a be the geodesic of strictly rank one. Then there exists a point $x_0 \in a$, a geodesic a' , connected with a by the asymptotic chain, and $\Delta > 0$, such that for every $\varepsilon \in (0, \Delta)$ there exist scissors $\langle a', b, c, d; x \rangle$, with displacement of scissors translation T equal to $\delta T = \varepsilon$.*

3.7. Recovery of the metric on geodesic of strictly rank one. The goal of this paragraph is to show that the metric of an arbitrary geodesic of strictly rank one is restorable from the diagonal tube V .

Lemma 3.34. *Let $x \notin a$ and $\xi, \eta \in \partial_\infty X$. Then the relation V allows to detect whether the triple (ξ, η, x) belongs to a set $Z(a)$.*

Proof. $(\xi, \eta, x) \in Z(a)$ if and only if there exists r -sequences $\{x_z\}_{z \in \mathbb{Z}} \subset a$, $\{u_z\}_{z \in \mathbb{Z}}$, $\{v_z\}_{z \in \mathbb{Z}}$ and $\{w_z\}_{z \in \mathbb{Z}}$, for which $u_0 = v_0 = x$, and such that their limiting points in $\partial_\infty X$ satisfy equalities $u_{-\infty} = x_{-\infty}$, $v_{+\infty} = x_{+\infty}$, $w_{-\infty} = v_{-\infty} = \eta$ and $w_{+\infty} = u_{+\infty} = \xi$. Given any four r -sequences $\{x_z\}_{z \in \mathbb{Z}}$, $\{u_z\}_{z \in \mathbb{Z}}$, $\{v_z\}_{z \in \mathbb{Z}}$ and $\{w_z\}_{z \in \mathbb{Z}}$, relation V detects, whether marked equalities hold or not. Also, V lets to reveal such four r -sequences if they exist. \square

Lemma 3.35. *Given scissors $\langle a, b, c, d; x \rangle$, relation V defines an image $T(m) \in a$ of any point $m \in a$ in scissors translation T .*

Proof. We use the fact that all horospheres are defined by V and serves as level sets of Busemann functions. Hence V defines a point $m_1 = R_{ac}(m)$ as $\mathcal{HS}_{a(+\infty), m} \cap c$, a point $m_2 = R_{cd}(m_1)$ as $\mathcal{HS}_{d(-\infty), m_1} \cap d$, a point $m_3 = R_{db}(m_2)$ as $\mathcal{HS}_{d(+\infty), m_2} \cap b$ and finally, a point $T(m) = m_4 = R_{ba}(m_3)$ as $\mathcal{HS}_{a(-\infty), m_3} \cap a$. \square

Lemma 3.36. *Let $\langle a, b, c, d; x \rangle$ be scissors with displacement of translation δT . Then V allows to display the value δT .*

Proof. We have:

$$\delta T = \frac{1}{n}d(x_0, T^n(x_0)) = \lim_{n \rightarrow +\infty} \frac{1}{n}d(x_0, T^n(x_0)) = \lim_{n \rightarrow +\infty} \frac{1}{n}[d(x_0, T^n(x_0))],$$

where $[t]$ is integral part of the number t . Since every item of the sequence $a_n = [d(x_0, T^n(x_0))]$ is definable by V , hence the value δT is. \square

We are ready now to prove the following theorem.

Theorem 3.37. *Let metric space (X, d) and trial metric d' on X be as in Theorem 0.4. Assume that $a : (-\infty, +\infty) \rightarrow X$ is a geodesic of strictly rank one in metric d . Then a is of strictly rank one in metric d' as well, and for all $t_1, t_2 \in \mathbb{R}$*

$$(3.38) \quad d'(c(t_1), c(t_2)) = d(c(t_1), c(t_2)).$$

Proof. We may think the assertion on the rank of a as already proved in Proposition 1.14, since the property of geodesic to have rank one is accomplished when any its r -sequence is of rank one.

According Corollary 3.33 there exists $x_0 \in a$, geodesic a' and $\Delta > 0$ such that for any $q \in \mathbb{N}$, satisfying $1/q < \Delta$, there exist scissors $\langle a', b, c, d; x \rangle$ with displacement $\delta T = 1/q$. By lemma 3.36 the relation V lets to detect, whether the value of displacement equal $1/q$. Take scissors with this value of displacement, which is independent on the choise of metric d or d' . We have

$$d(a'(t), T^p(a'(t))) = d'(a'(t), T^p(a'(t))) = p/q$$

for all $t \in \mathbb{R}$. Described procedure defines all points of type $a'(t)$ with $t \in \mathbb{Q}$ in a' from relation V , and we have $d(a'(0), a'(t)) = d'(a'(0), a'(t))$ for them.

When $t \in \mathbb{R} \setminus \mathbb{Q}$ we may use the fact that V defines the incidence relation for a' and its order of points.

It remains to apply remark 3.23 and Lemma 1.10 for completing the proof. \square

Combining Theorem 3.37 with Theorem 2.14, one gets the whole proof of the Theorem 0.4. Consequently Theorem 0.1 is also proved.

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